Grade 11/12 Math Circles
October 11 2023
Digital Signal Processing

Representing Digital Signals Using the Delta Function

Last session we introduced a special signal called the *delta function*, also known as *the unit impulse*. Recall that the delta function is denoted by $\delta[n]$, and is defined as:

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

The delta function looks like a spike of height one at $n = 0$:

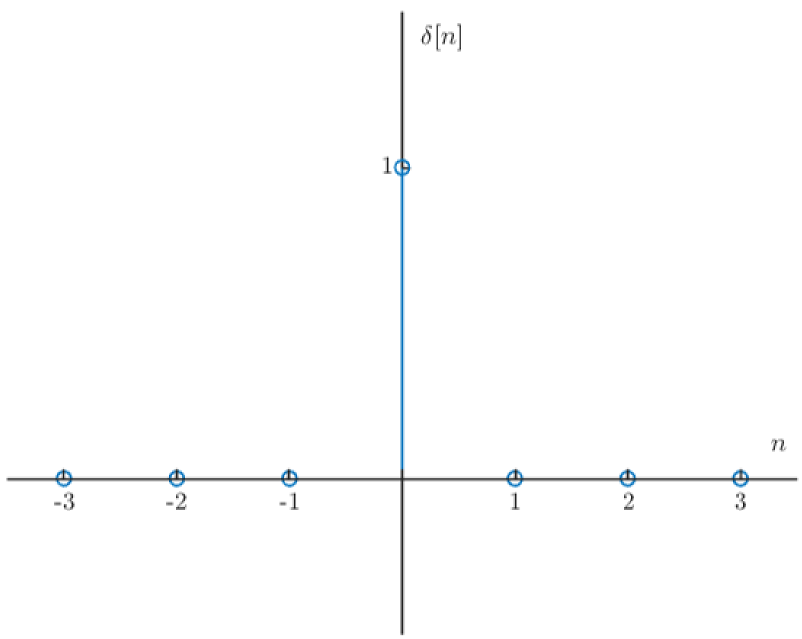


Figure 1: The delta function (unit impulse), $\delta[n]$.

The delta function has a number of nice properties which make it useful in the study of digital signal processing, including the fact that we can write any digital signal as a weighted sum of shifted delta functions.

Before we do this, let's introduce some (maybe new) notation for writing sums!

Summation (Sigma) Notation

The symbol $\sum$ (sigma) is used to represent sums. For example, the sum

$$1+2+3+4+5+6+7+8+9+10=\sum_{k=1}^{10} k.$$

We read this as "the sum from $k=1$ to $k=10$ of k". The variable k is called the *summation index* and takes on all integer values from 1 to 10.

Using sigma notation we have a nice way to represent infinite sums as well, for instance

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots=\sum_{k=1}^{\infty}\frac{1}{k}.$$

Now, let's get some practice using sigma notation.

Exercise 1

Evaluate the following sums:

a) $\sum_{k=1}^{4} k^2$

b) $\sum_{k=1}^{5}(2k+1)$

c) $\sum_{n=3}^{5} n(n+1)$

d) $\sum_{j=-1}^{1}(j+1)$

Now that we've gotten some practice using sigma notation, let's see how we can use this to represent digital signals in terms of shifted delta functions. Next session, we will see why this is very useful when working with digital filters.

Consider the signal $x[n]$ which is defined by the following sequence of values (and assumed to be equal to zero everywhere else):

n	0	1	2	3	4
$x[n]$	1	3	5	-1	2

The signal $x[n]$ can also be represented by the following plot which makes it clear that $x[n]$ is made up of spikes of varying heights at each value of n:

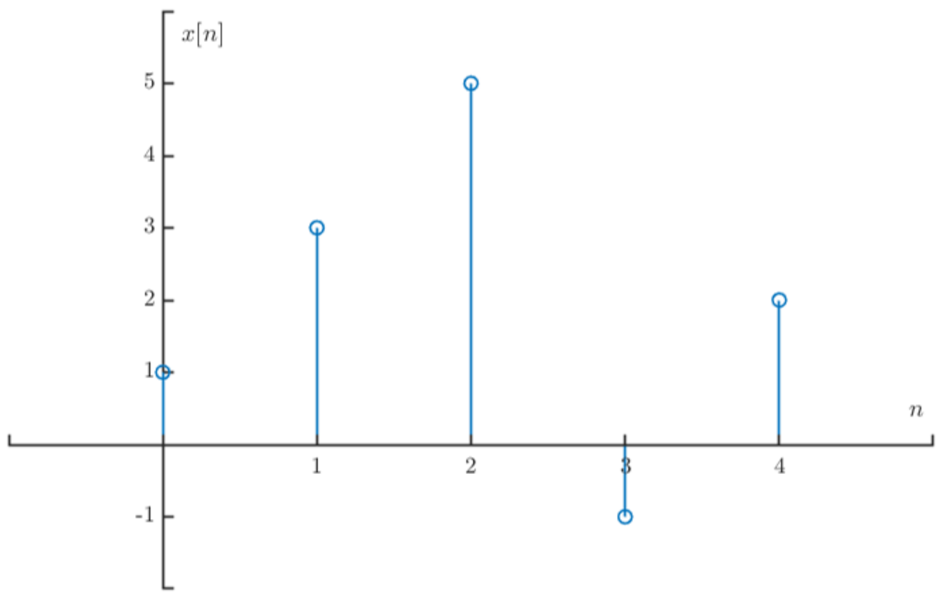


Figure 2: The signal $x[n]$.

As a result, we can construct this signal by adding together shifted delta functions, as shown below:

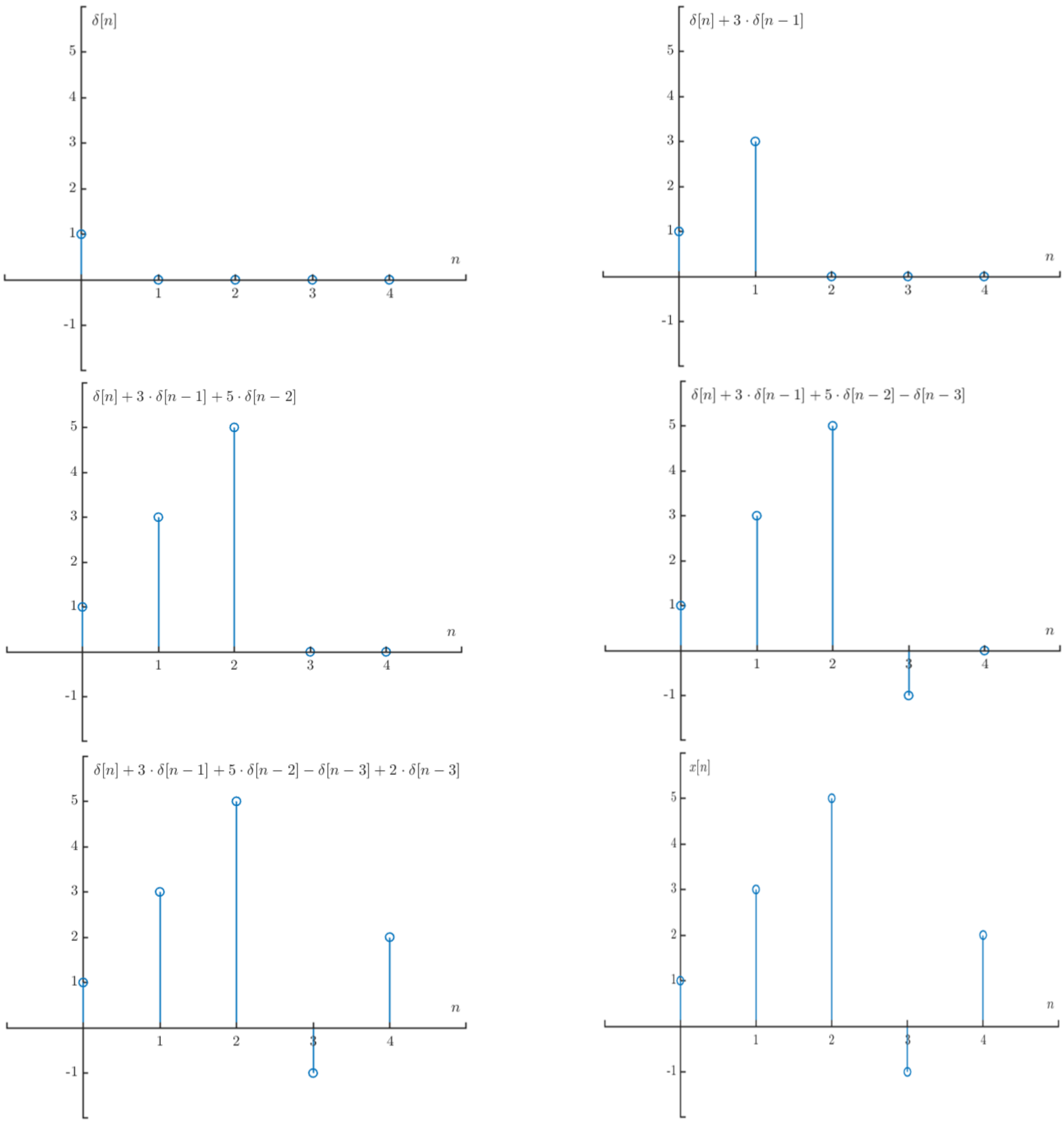


Figure 3: Constructing $x[n]$ using shifted delta functions.

We see that the signal $x[n]$ can be written as

$$x[n] = 1 \cdot \delta[n] + 3 \cdot \delta[n-1] + 5 \cdot \delta[n-2] + (-1) \cdot \delta[n-3] + 2 \cdot \delta[n-4].$$

In general we can write any signal $x[n]$ as

$$x[n] \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Let's get some practice!

Exercise 2

Write each of the following finite signals as a weighted sum of shifted delta functions.

a)

n	0	1	2	3	4
$x[n]$	1	3	5	-1	2

b) $x[n] = \begin{cases} 0 & \text{if } n < 0 \\ n & \text{if } 0 \leq n \leq 3 \\ 0 & \text{if } n > 3 \end{cases}$

For an extra challenge: see if you can write this using sigma ($\sum$) notation!

The Impulse Response

Last session we introduced digital filters, which are systems which perform operations on digital signals, usually to reduce or enhance certain aspects of that signal. Digital filters take a signal as an input, i.e. $x[n]$, and produce a modified signal as an output, i.e. $y[n]$. This is often represented using *block diagrams*, like the one shown below.

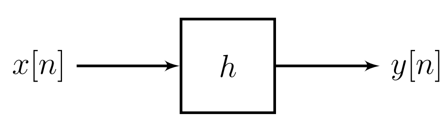


Figure 4: The signal $x[n]$ is processed using filter h to produce the output signal $y[n]$.

Certain types of digital filters can be completely characterized by their *impulse response*. The impulse response of a digital filter is the output of the filter when the input signal is the delta function (or unit impulse). This is usually denoted by $h[n]$, which itself is a digital signal.

Example 1

Recall that the moving average filter with window size N is defined by:

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k].$$

where $x[n]$ is the input signal and $y[n]$ is the output signal.

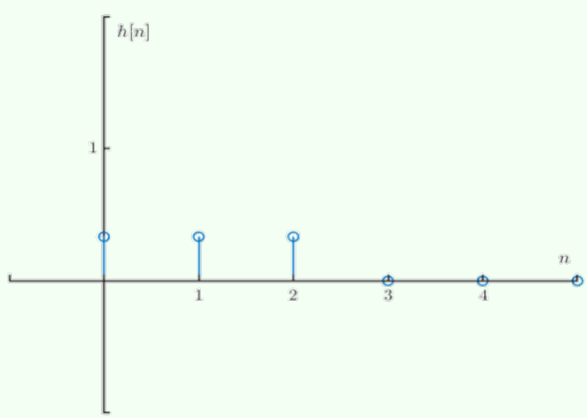
To determine the impulse response we let the input signal be $\delta[n]$, so that

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} \delta[n-k].$$

Since $\delta[n-k]$ is only non-zero when $n=k$, we see that

$$h[n] = \begin{cases} \frac{1}{N} & \text{if } 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $N=3$, the impulse response, $h[n]$ looks like:



We can determine the impulse response of any digital filter by evaluating the output of the filter when the input signal is the delta function. Let's give it a try!

Exercise 3

Determine the impulse response of the following digital filters.

a) $y[n] = x[n] - x[n-1]$

b) $y[n] = \text{median}\,(x[n], x[n-1], x[n-2])$

The first filter is an example of what is called a *linear* filter, which can be completely characterized by its impulse response. On the other hand, the median filter is an example of a *non-linear* filter which is **not** completely characterized by its impulse response.

Properties of Digital Filters

Linearity (and non-linearity) is one example of how we can classify digital filters. Let's discuss the three main ways in which digital filters are classified.

Causality: A digital filter is *causal* if the output at time n only depends on the input up to time n. For example, the filter defined by

$$y[n] = x[n] - 2 \cdot x[n-1]$$

is a causal filter, whereas the filter defined by

$$y[n] = x[n+3]$$

is not causal because $y[n]$ depends on the input at time $n+3$. Filters which operate on signals in real time are always causal.

Exercise 4

Determine which of the following digital filters are causal.

a) $y[n] = \sum_{k=1}^{N} k \cdot x[n-k]$

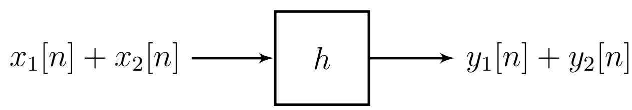
b) $y[n] = \text{median}\,(x[n], x[n-1], x[n-2])$

c) $y[n] = (x[n])^2$

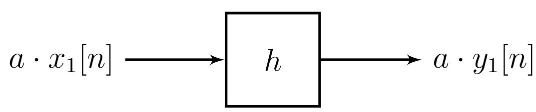
d) $y[n] = \frac{x[n]+x[n+1]}{2}$

Linearity: A linear digital filter has the following two properties:

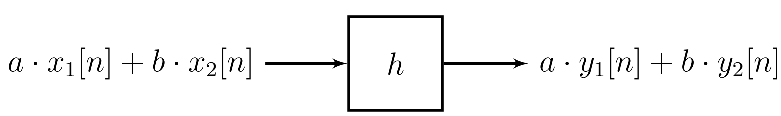
1. Additivity:



2. Homogeneity:



where a is a constant, $x_1[n]$ is a signal with corresponding output $y_1[n]$ and $x_2[n]$ is a signal with corresponding output $y_2[n]$. Combining these two properties, we say that a digital filter is linear if it satisfies the following:



for *any* input signals $x_1[n]$ and $x_2[n]$ and any constants a and b.

Example 2

Show that the filter defined by

$$y[n] = x[n] - 2 \cdot x[n-1].$$

is linear.

To show that this is a linear filter, first we define two input signals $x_1[n]$ and $x_2[n]$ which have outputs

$$\begin{aligned} x_1[n] &\rightarrow x_1[n] - 2 \cdot x_1[n-1] = y_1[n] \\ x_2[n] &\rightarrow x_2[n] - 2 \cdot x_2[n-1] = y_2[n]. \end{aligned}$$

Now, let $z[n] = a \cdot x_1[n] + b \cdot x_2[n]$ be the input to the filter. The output is

$$\begin{aligned} z[n] - 2 \cdot z[n-1] &= a \cdot x_1[n] + b \cdot x_2[n] - 2(a \cdot x_1[n-1] + b \cdot x_2[n-1]) \\ &= a(x_1[n] - 2 \cdot x_1[n-1]) + b(x_2[n] - 2 \cdot x_2[n-1]) \\ &= a \cdot y_1[n] + b \cdot y_2[n]. \end{aligned}$$

Therefore, this filter is linear.

As we just saw, in order to show that a filter is linear, we need to show that both the additivity and homogeneity properties apply for all signals $x_1[n]$ and $x_2[n]$.

On the other hand, to show that a filter is non-linear, we just need to find at least one example where either property (or both properties) does not apply. This is called finding a *counterexample*.

Example 3

Consider the filter defined by

$$y[n] = (x[n])^2 .$$

Let $x_1[n] = 1$ (i.e. a signal which is equal to 1 everywhere) and $x_2[n] = 2$ (i.e. a signal which is equal to 2 everywhere). The corresponding outputs are

$$\begin{aligned} y_1[n] &= (x_1[n])^2 = 1 \\ y_2[n] &= (x_2[n])^2 = 4. \end{aligned}$$

Now, let $z[n] = x_1[n] + x_2[n]$. With $z[n]$ as input, the output of the filter is

$$\begin{aligned} (z[n])^2 &= (x_1[n] + x_2[n])^2 \\ &= (1+2)^2 \\ &= 3^2 \\ &= 9 \neq y_1[n] + y_2[n] = 1 + 4 = 5. \end{aligned}$$

As a result, this filter is not linear.

Now it's your turn!

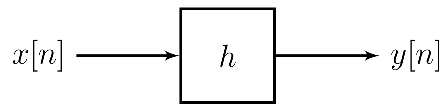
Exercise 5

Find a counterexample to show that the following filters are non-linear.

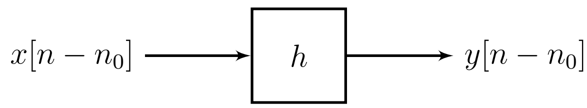
a) $y[n] = 3x[n] + 5$

b) **CHALLENGE:** $y[n] = \text{median}\left(x[n], x[n-1], x[n-2]\right)$

Time-Invariance: A filter is said time-invariant if it behaves the same way, regardless of when the input is applied. This means that if we have



then we also have



i.e. if we delay the input signal by some amount of time n_0, we get the same output, just delayed (or shifted) by n_0. In order for a filter to be time-invariant, this needs to be true for all input signals $x[n]$ and all time delays n_0.

Example 4

Once again, consider the filter defined by

$$y[n] = x[n] - 2 \cdot x[n-1].$$

Let's check if this filter is time-invariant.

Let $z[n] = x[n-n_0]$. When $z[n]$ is the input, the filter output is

$$\begin{aligned} z[n] - 2 \cdot z[n-1] &= x[n-n_0] - 2 \cdot x[n-1-n_0] \\ &= x[n-n_0] - 2 \cdot x[n-n_0-1] \\ &= y[n-n_0] \end{aligned}$$

Therefore the filter is time-invariant.

Now it's your turn!

Exercise 6

Consider an input signal $x[n]$ and a time delayed signal $x[n-n_0]$. Use this to determine whether or not the following digital filters are time-invariant.

a) $y[n] = \frac{1}{2}(x[n] + x[n-1])$

b) $y[n] = x[n^2]$